

# Quasiperiodic Envelope Solitons

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We analyse nonlinear wave propagation and cascaded self-focusing due to second-harmonic generation in *Fibonacci optical superlattices* and introduce a novel concept of nonlinear physics, the *quasiperiodic soliton*, which describes spatially localized self-trapping of a quasiperiodic wave. We point out a link between the quasiperiodic soliton and partially incoherent spatial solitary waves recently generated experimentally.

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For many years, solitary waves (or *solitons*) have been considered as *coherent localized modes* of nonlinear systems, with particle-like dynamics quite dissimilar to the irregular and stochastic behaviour observed for chaotic systems [1]. However, about 20 years ago Akira Hasegawa, while developing a statistical description of the dynamics of an ensemble of plane waves in nonlinear strongly dispersive plasmas, suggested the concept of an *incoherent temporal soliton*, a localised envelope of random phase waves [2]. Because of the relatively high powers required for generating self-localised random waves, this notion remained a theoretical curiosity until recently, when the possibility to generate spatial optical solitons by a partially incoherent source was discovered in a photorefractive medium [3], known to exhibit strong nonlinear effects at low powers.

The concept of incoherent solitons can be compared with a different problem: the propagation of a soliton through a spatially disordered medium. Indeed, due to random scattering on defects, the phases of the individual components forming a soliton experience random fluctuations, and the soliton itself becomes *partially incoherent* in space and time. For a low-amplitude wave (linear regime) spatial incoherence is known to lead to a fast decay. As a result, the transmission coefficient vanishes exponentially with the length of the system, the phenomenon known as Anderson localisation [4]. However, for large amplitudes (nonlinear regime), when the nonlinearity length is much smaller than the Anderson localization length, a soliton can propagate almost unchanged through a disordered medium as predicted theoretically in 1990 [5] and recently verified experimentally [6].

These two important physical concepts, spatial self-trapping of light generated by an incoherent source in a homogeneous medium, and suppression of Anderson localisation for large-amplitude waves in spatially disordered media, both result from the effect of strong nonlinearity. When the nonlinearity is sufficiently strong it acts as an *effective phase-locking mechanism* by producing a large frequency shift of the different random-phase com-

ponents, and thereby introducing an *effective order* into an incoherent wave packet, thus enabling the formation of localised structures. In other words, both phenomena correspond to the limit when the ratio of the nonlinearity length to the characteristic length of (spatial or temporal) fluctuations is small. In the opposite limit when this ratio is large the wave propagation is basically linear.

*What will happen in the intermediate case when the length scales of nonlinearity and fluctuations become comparable ?* It is usually believed that localised structures would not be able to survive for such incoherent wave propagation and should rapidly decay. In this Letter we show that, at least for aperiodic inhomogeneous structures, solitary waves can exist in the form of *quasiperiodic nonlinear localised modes*. As an example we consider second-harmonic generation (SHG) and nonlinear beam propagation in *Fibonacci optical superlattices*, and demonstrate numerically the possibility of spatial self-trapping of quasiperiodic waves whose envelope amplitude varies quasiperiodically, while still maintaining a stable, well-defined spatially localised structure, a *quasiperiodic envelope soliton*.

We consider the interaction of a fundamental wave (FW) with the frequency  $\omega$  and its second harmonic (SH) in a slab waveguide with quadratic (or  $\chi^{(2)}$ ) nonlinearity. Assuming the  $\chi^{(2)}$  susceptibility to be modulated and the nonlinearity to be of the same order as diffraction, we write the dynamical equations in the form

$$\begin{aligned} i \frac{\partial w}{\partial z} + \frac{1}{2} \frac{\partial^2 w}{\partial x^2} + d(z) w^* v e^{-i\beta z} &= 0, \\ i \frac{\partial v}{\partial z} + \frac{1}{4} \frac{\partial^2 v}{\partial x^2} + d(z) w^2 e^{i\beta z} &= 0, \end{aligned} \quad (1)$$

where  $w(x, z)$  and  $v(x, z)$  are the slowly varying envelopes of the FW and SH, respectively. The parameter  $\beta = \Delta k |k_\omega| x_0^2$  is proportional to the phase mismatch  $\Delta k = 2k_\omega - k_{2\omega}$ ,  $k_\omega$  and  $k_{2\omega}$  being the wave numbers at the two frequencies. The transverse coordinate  $x$  is measured in units of the input beam width  $x_0$ , and the propagation distance  $z$  in units of the diffraction length

$l_d = x_0^2 |k_\omega|$ . The spatial modulation of the  $\chi^{(2)}$  susceptibility is described by the quasi-phase-matching (QPM) grating function  $d(z)$ . In the context of SHG, the QPM technique is an effective way to achieve phase matching, and has been studied intensively (see Ref. [7] for a comprehensive review).

Here we consider a QPM grating produced by a quasiperiodic nonlinear optical superlattice. Quasiperiodic optical superlattices, one-dimensional analogs of quasicrystals [8], are usually designed to study the effect of Anderson localisation in the linear regime of light propagation. For example, Gellermann *et al.* measured the optical transmission properties of quasiperiodic dielectric multilayer stacks of SiO<sub>2</sub> and TiO<sub>2</sub> thin films and observed a strong suppression of the transmission [9]. For QPM gratings, a nonlinear quasiperiodic superlattice of LiTaO<sub>3</sub>, in which two antiparallel ferro-electric domains are arranged in a Fibonacci sequence, was recently fabricated by Zhu *et al.* [10], who measured multi-colour SHG with energy conversion efficiencies of  $\sim 5\% - 20\%$ . This quasiperiodic optical superlattice in LiTaO<sub>3</sub> can also be used for efficient direct third harmonic generation [11].

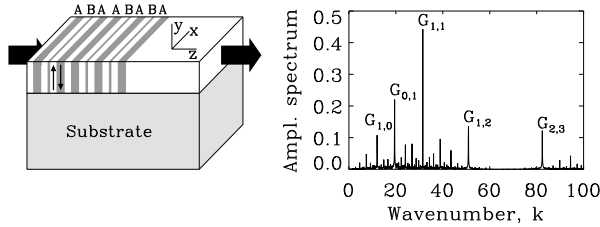


FIG. 1. (a) Slab waveguide with quasiperiodic QPM superlattice structure composed of building blocks A and B. (b) Numerically calculated amplitude spectrum of  $d(z)$ .

The quasiperiodic QPM gratings have two building blocks A and B of the length  $l_A$  and  $l_B$ , respectively, which are ordered in a Fibonacci sequence [Fig. 1(a)]. Each block has a domain of length  $l_{A1}=l$  ( $l_{B1}=l$ ) with  $d=+1$  (shaded) and a domain of length  $l_{A2}=l(1+\eta)$  [ $l_{B2}=l(1-\tau\eta)$ ] with  $d=-1$  (white). In the case of  $\chi^{(2)}$  nonlinear QPM superlattices this corresponds to positive and negative ferro-electric domains, respectively. The specific details of this type of Fibonacci optical superlattices can be found elsewhere (see, e.g., Ref. [10] and references therein). For our simulations presented below we have chosen  $\eta = 2(\tau - 1)/(1 + \tau^2) = 0.34$ , where  $\tau = (1 + \sqrt{5})/2$  is the so-called *golden ratio*. This means that the ratio of length scales is also the golden ratio,  $l_A/l_B = \tau$ . Furthermore, we have chosen  $l=0.1$ .

The grating function  $d(z)$ , which varies between  $+1$  and  $-1$  according to the Fibonacci sequence, can be expanded in a Fourier series

$$d(z) = \sum_{m,n} d_{m,n} e^{iG_{m,n}z}, \quad G_{m,n} = \frac{2\pi(m + n\tau)}{D}, \quad (2)$$

where  $D = \tau l_A + l_B = 0.52$  for the chosen parameter values. Hence the spectrum is composed of sums and differences of the basic wavenumbers  $\kappa_1 = 2\pi/D$  and  $\kappa_2 = 2\pi\tau/D$ . These components fill the whole Fourier space densely, since  $\kappa_1$  and  $\kappa_2$  are incommensurate. Figure 1(b) shows the numerically calculated Fourier spectrum  $G_{m,n}$ . The lowest-order “Fibonacci modes” are clearly the most intense. From Eq. (2) and the numerically found spectrum we identify the six most intense modes presented in Table 1. The corresponding wavenumbers  $G_{m,n}$  are in good agreement with Eq. (2).

$m$	1	0	1	2	1	2
$n$	1	1	2	3	0	4
$G_{m,n}$	31.42	19.42	50.83	82.25	12.00	101.66

TABLE 1. The six most intense Fibonacci modes  $G_{m,n}$ .

To analyse the beam propagation and SHG in a quasiperiodic QPM grating one could simply average Eqs. (1). To lowest order this approach always yields a system of equations with constant mean-value coefficients, which does not allow to describe oscillations of the beam amplitude and phase. However, here we wish to go beyond the averaged equations and consider the rapid large-amplitude variations of the envelope functions. This can be done analytically for periodic QPM gratings [12]. However, for the quasiperiodic gratings we have to resort to numerical simulations.

Thus we have solved Eqs. (1) numerically with a second-order split-step routine, in which the linear part is solved with the fast-Fourier-transform (FFT) method and the nonlinear part, with a fourth-order Runge-Kutta scheme. The step-length is adapting to the local domain length of the QPM grating. At the input of the crystal we excite the fundamental beam (corresponding to unseeded SHG) with a Gaussian profile,

$$w(x, 0) = A_w e^{-x^2/10}, \quad v(x, 0) = 0. \quad (3)$$

We consider the quasiperiodic QPM grating with matching to the peak at  $G_{2,3}$ , i.e.,  $\beta = G_{2,3} = 82.25$ . First, we study the small-amplitude limit when a weak FW is injected with a low amplitude. Figures 2(a,b) show an example of the evolution of FW and SH in this effectively linear regime. As is clearly seen from Fig. 2(b) the SH wave is excited, but both beams eventually diffract.

When the amplitude of the input beam exceeds a certain threshold, self-focusing and localization should be observed for both harmonics. Figures 2(c,d) show an example of the evolution of a strong input FW beam, and its corresponding SH. Again the SH is generated, but now the nonlinearity is so strong that it leads to self-focusing and mutual self-trapping of the two fields, resulting in a spatially localized two-component soliton, despite the continuous scattering of the quasiperiodic QPM grating.

It is important to notice that the two-component localised beam created due to the self-trapping effect is

quasiperiodic by itself. As a matter of fact, after an initial transient its amplitude oscillates in phase with the quasiperiodic QPM modulation  $d(z)$ . This is illustrated in Fig. 3, where we show in more detail the peak intensities in the asymptotic regime of the evolution.

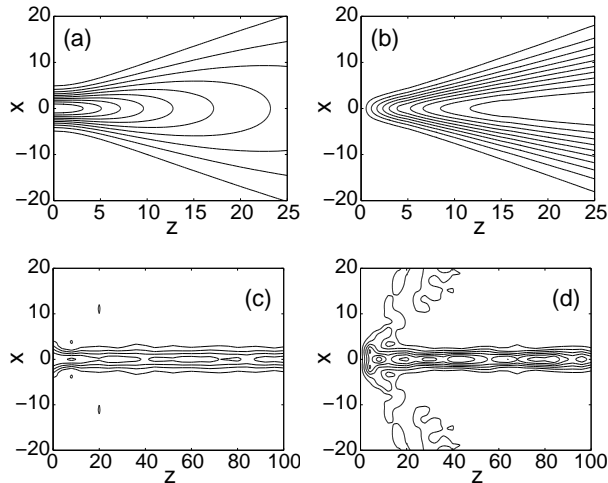


FIG. 2. (a) Diffraction of a weak FW beam with amplitude  $A_w=0.25$  for  $\beta=82.25$ . (c) Excitation of a quasiperiodic soliton by a FW beam with amplitude  $A_w=5$  for  $\beta=82.25$ . (b,d) Corresponding SH components.

Since the oscillations shown in Fig. 3 are in phase with the oscillations of the QPM grating  $d(z)$ , their spectra should be similar. This is confirmed by Fig. 4, which gives the spectrum of the peak intensity  $|w(z,0)|^2$  of the FW. Note that the Fibonacci peak at  $k=82.25$  is suppressed (or reduced) because the identical mismatch  $\beta$  down-converts it to the dc-component. Sum and difference wavenumbers between  $\beta$  and  $G_{m,n}$  appear, which are generated by the nonlinearity. For example, the component at  $k=62.8$  is the difference between  $\beta=82.25$  and  $G_{0,1}=19.42$ .

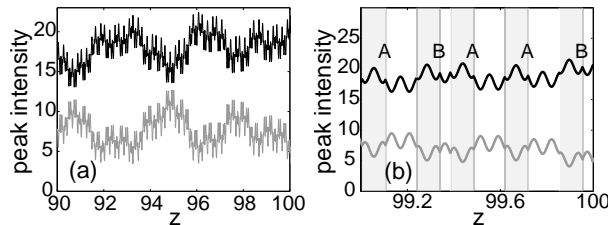


FIG. 3. Amplitude oscillations of the quasiperiodic soliton. (a),(b) Close-ups of the peak intensity  $|w(0,z)|^2$  of the FW (black) and  $|v(0,z)|^2$  of the SH (grey). The Fibonacci building blocks A and B are indicated in (b) with  $d=1$  in grey regions, and  $d=-1$  in white regions.

Our numerical results show that the quasiperiodic envelope solitons can be generated for a broad range of the phase-mismatch  $\beta$ . The amplitude and width of the solitons depend on the effective mismatch, which is the separation between  $\beta$  and the nearest strong peak  $G_{m,n}$  in the

Fibonacci QPM grating spectrum. Thus, low-amplitude broad solitons are excited for  $\beta$ -values in between peaks, whereas high-amplitude narrow solitons are excited when  $\beta$  is close to a strong peak, as shown in Fig. 2(c,d).

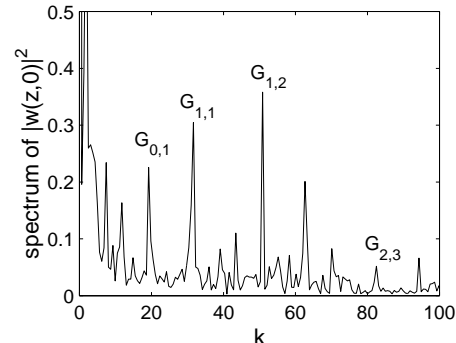


FIG. 4. Spectrum of the amplitude oscillations of the FW component of the quasiperiodic soliton, calculated from  $z=90$  to 100 in Fig. 3(a). The peaks correspond to the Fibonacci peaks  $G_{m,n}$  in  $d(z)$  and sum and difference thereof with the mismatch  $\beta=82.25$ .

The existence of spatially localized self-trapped states in nonlinear quasiperiodic media should not depend on the particular kind of nonlinearity. The dependence on  $\beta$  observed here for the  $\chi^{(2)}$  gratings is simply because the "real" strength of the  $\chi^{(2)}$  nonlinearity is inversely proportional to the phase-mismatch. In fact, it is well-known that for large values of the mismatch  $\beta$  the quadratic nonlinearity becomes effectively cubic [13]. Thus, our findings are directly applicable to nonlinear optical superlattices in cubic (or  $\chi^{(3)}$ ) nonlinear media.

To analyse in more detail the transition between the linear (diffraction) and nonlinear (self-trapping) regimes, we have made a series of careful numerical simulations. In Fig. 5 we show the transmission coefficients and the beam widths at the output of the crystal versus the intensity of the FW input beam, for a variety of  $\beta$ -values. These dependencies clearly illustrates the universality of the generation of localised modes for varying strength of nonlinearity, i.e. a quasiperiodic soliton is generated only for sufficiently high amplitudes. This is of course a general phenomenon also observed in many nonlinear isotropic media. However, here the self-trapping occurs for quasiperiodic waves, with the quasiperiodicity being preserved in the variation of the amplitude of both components of the soliton.

Numerical simulations for other values of the phase-mismatch  $\beta$  reveal the same basic property of quasiperiodic self-trapping: Spatial solitons are formed in Fibonacci quadratic nonlinear slab waveguides above a certain power threshold, and such solitons are always *quasiperiodic*, i.e. they exhibit large-amplitude oscillations along  $z$ , which are composed of mixing of the two incommensurate Fibonacci wavenumbers  $\kappa_1$  and  $\kappa_2$ . The amplitude and width of these solitons depend on the dif-

ference between the phase-mismatch  $\beta$  and the nearest strong peak  $G_{m,n}$  in the Fibonacci spectrum.

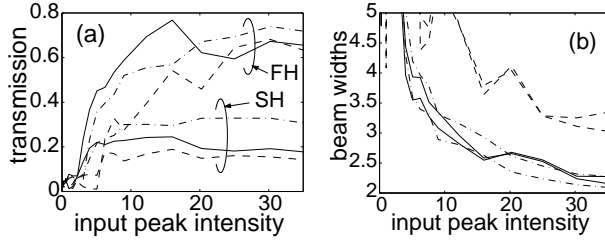


FIG. 5. (a) Transmission of the FW,  $|w(0, L)/w(0, 0)|^2$ , and the SH,  $|v(0, L)/w(0, 0)|^2$  vs. input peak intensity  $|w(0, 0)|^2$  of the FW. (b) Output beam width vs.  $|w(0, 0)|^2$ .  $L=100$ ,  $\beta=G_{1,2}=50.83$  (solid),  $\beta=G_{2,4}=101.66$  (dashed), and  $\beta=G_{2,3}=82.25$  (dot-dashed).

Finally we would like to emphasize that the phenomenon described here is qualitatively different from the propagation of topological and nontopological *kinks* in disordered and quasiperiodic nonlinear media [4]. Such kinks can be well approximated by an effective structureless particle, which either preserves identity, as in the case of topological kinks [14,15], or decays rapidly into radiation [16].

In conclusion, we have analysed SHG, self-focusing, and nonlinear beam propagation in Fibonacci optical superlattices with a quadratic nonlinear response. We have predicted spatial self-trapping of quasiperiodic waves and the formation of quasiperiodic solitons. Such solitons have a localised envelope that traps the random-phase components through the phase and frequency locking effect of strong nonlinearity, and whose amplitude undergoes clearly detectable quasiperiodic oscillations. The results presented here would allow to extend the concepts of self-localisation and self-modulation of nonlinear waves to a broader class of spatially inhomogeneous media, and can also be found in systems of different physical context.

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- [1] See, e.g., “*Future Directions of Nonlinear Dynamics in Physical and Biological Systems*”, P.L. Christiansen *et al.*, Eds., NATO ASI Series B: Physics, Vol. 312 (Plenum Press, New York, 1993).
  - [2] A. Hasegawa, Phys. Fluids **18**, 77 (1975); Phys. Fluids **20**, 2155 (1977).
  - [3] M. Mitchell *et al.*, Phys. Rev. Lett. **77**, 490 (1996); see also M. Mitchell and M. Segev, Nature **387**, 880 (1997); Z. Chen *et al.*, Science **280**, 889 (1998).
  - [4] See, e.g., S.A. Gredeckul and Yu.S. Kivshar, Phys. Rep. **216**, 1 (1992), and references therein.
  - [5] Yu.S. Kivshar, S.A. Gredeckul, A. Sánchez, and L. Vázquez, Phys. Rev. Lett. **64**, 1693 (1990).
  - [6] V.A. Hopkins, J. Keat, G.D. Meegan, T. Zhang, and J.D. Maynard, Phys. Rev. Lett. **76**, 1102 (1996).
  - [7] M.M. Fejer, G.A. Magel, D.H. Jundt, and R.L. Byer, IEEE J. Quantum Electron. **28**, 2631 (1992).
  - [8] D. Schechtman, I. Blech, D. Gratias, and J.W. Cahn, Phys. Rev. Lett. **53**, 1951 (1984).
  - [9] W. Gellermann, M. Kohmoto, B. Sutherland, and P.C. Taylor, Phys. Rev. Lett. **72**, 63 (1994).
  - [10] S. Zhu, Y. Zhu, Y. Qin, H. Wang, C. Ge, and N. Ming, Phys. Rev. Lett. **78**, 26752 (1997).
  - [11] S. Zhu, Y. Zhu, and N. Ming, Science **278**, 843 (1997).
  - [12] C. Balslev Clausen, O. Bang, and Yu.S. Kivshar, Phys. Rev. Lett. **78**, 4749 (1997).
  - [13] See, e.g., Yu.S. Kivshar, In: *Advanced Photonics with Second-order Optically Nonlinear Processes*, Eds. A.D. Boardman *et al.* (Kluwer, Amsterdam, 1999), p. 451.
  - [14] S.A. Gredeckul *et al.*, Phys. Rev. A **45**, 8867 (1992).
  - [15] F. Dominguez-Adame, A. Sánchez, and Yu.S. Kivshar, Phys. Rev. E **52**, 1283 (1995).
  - [16] See, e.g., M. Hörnquist and R. Riklund, J. Phys. Soc. Jpn. **65**, 2872 (1996).